LECTURE 10

SIMULATION AND MODELING

CSE 411

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Confidence Interval : Going reverse

- Suppose the population mean is μ and population variance σ^2
  - What if we collect 10 normal random variable with mean $\mu = 2$ and variance $\sigma^2 = 1$

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[MATLAB Code]
MU = ones(1,10)*2;
SIGMA = ones(1,10);
R = normrnd(MU,SIGMA)
```

For example the sample is :

2.7143	3.6236	1.3082	2.8580	3.2540
0.4063	0.5590	2.5711	1.6001	2.6900

What does it mean : Going reverse

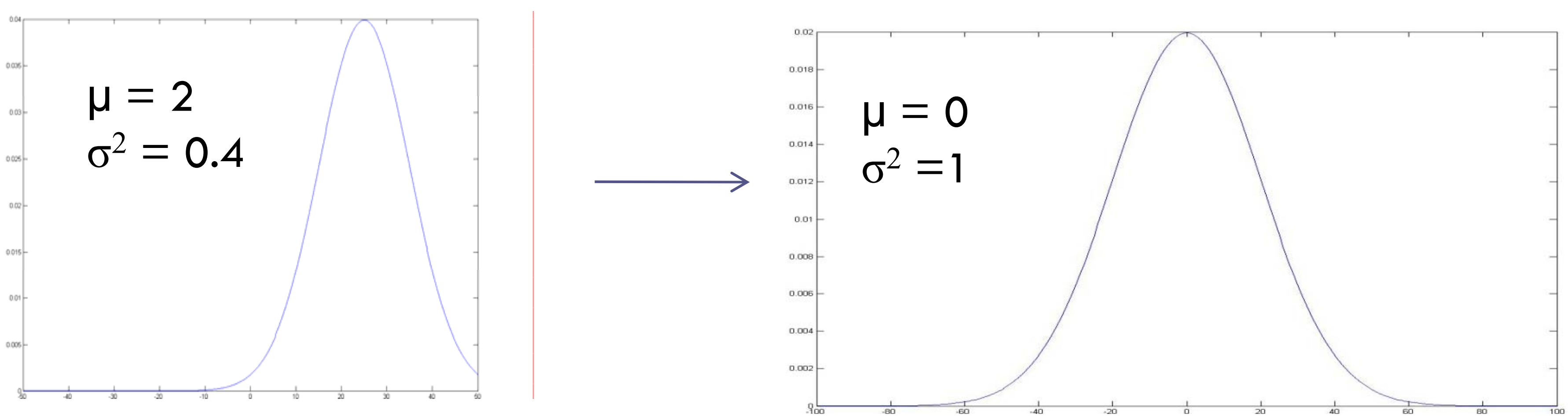
- What is the apriori probability that the sample mean is between a and b ??
- Suppose F(x) denotes the distribution function for sample mean, and f(x) be the density
- Then the probability that the sample mean is between a and b is :

$$\int_a^b f(x)dx = F(b) - F(a)$$

- Now, what is f ?? Without knowing the distribution, we can't proceed …

A shed of light …

- Central Limit theorem says that the distribution is normal !!!
  - For any set of IID numbers, picked from **whatever** distribution, the sample mean is a **normal random variable !!!**
  - Suppose we know μ and σ^2. Then we can convert the normal random variable to a Standard normal variable



A shed of light …

- Suppose $\overline{X}(n)$ is the sample mean.
- To transform it to a standard normal random variable, we use :

$$Z_n = \frac{\overline{X}(n) - \mu}{\sqrt{\frac{\sigma^2}{n}}} \qquad t_n = \frac{\overline{X}(n) - \mu}{\sqrt{\frac{S^2(n)}{n}}}$$

- Now we know f ….

$$\int_a^b f(x)dx = F(a) - F(b)$$

$$F(a) = \frac{1}{2\pi}\int_{-\infty}^{a} e^{-\frac{y^2}{2}} dy \qquad F(b) = \frac{1}{2\pi}\int_{-\infty}^{b} e^{-\frac{y^2}{2}} dy$$

Not analytically integrable ….

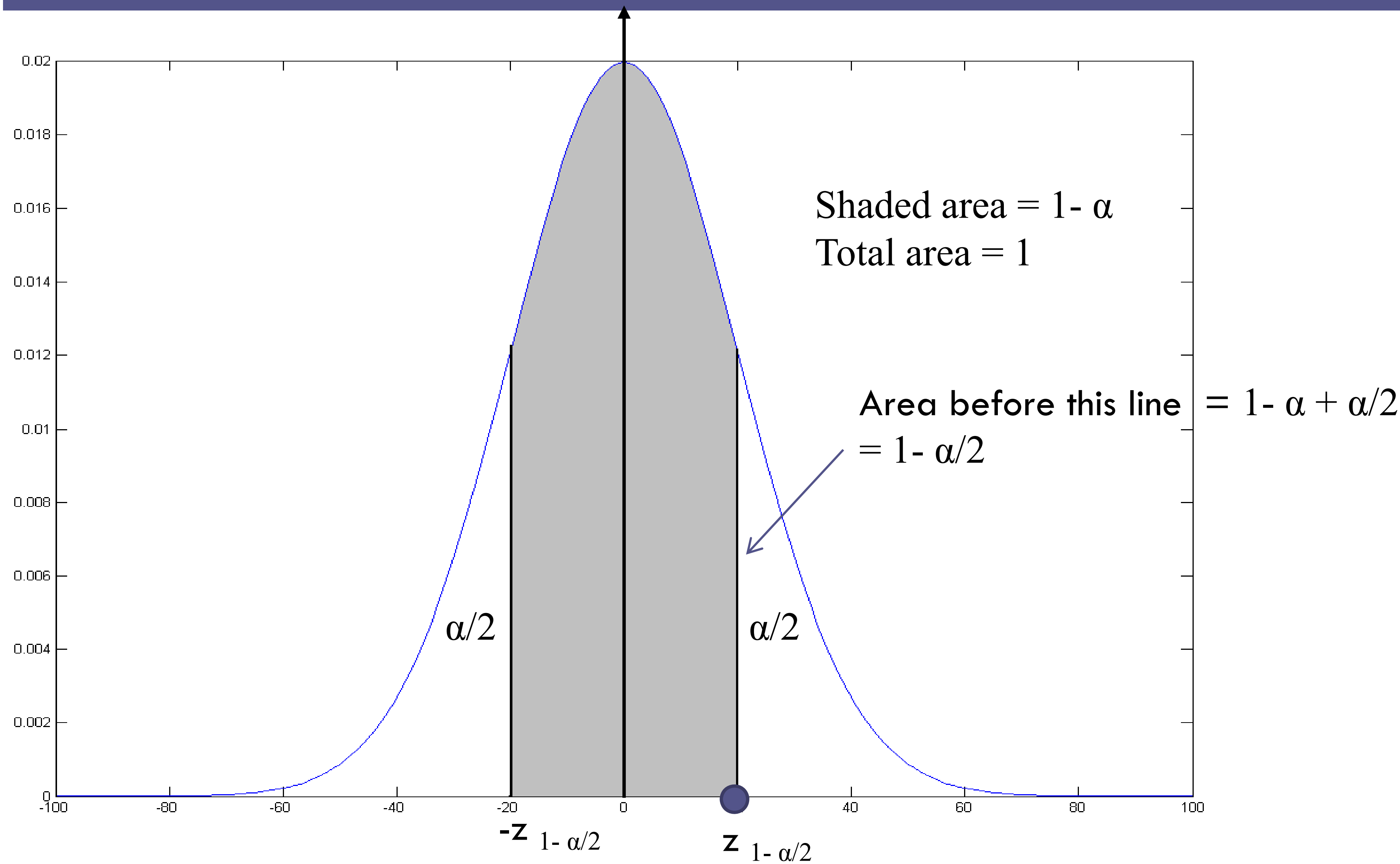
A shed of light …

- Remember confidence ? When we are 100(1-α) percent confident ?
  - When the probability is 1-α ….
  - So select two symmetric values such that the shaded area is 1-α. i.e.

$$\int_{-a}^{a} f(x)dx = F(a) - F(-a) = 1 - \alpha$$

  - So, given specific α we can find a.
    - Supppose that value is $z_{1-\alpha/2}$
    - So, given α, the left and right limits $z_{1-\alpha/2}$ $-z_{1-\alpha/2}$

The Confidence Interval



The Normal Probability Distribution

The meaning

- If one constructs a very large number of independent 100(1-α) percent confidence intervals, each based on n observations, where n is sufficiently large, the proportion of these confidence intervals that contain (cover) μ should be (1-α)

- We call this proportion the **coverage** for the confidence interval.
- Remember **coverage : 1-α**

Some difficulty

- The more skewed the underlying distribution of the Xi's,
  - the larger the value of n needed for the distribution of t_n to closely approximate a standard normal random variable.
- If n is chosen small
  - The actual coverage becomes less…

An alternate …

- **If Xi s are normal** random variable, t_n has a t distribution with n-1 dof

$$t_n = \frac{\overline{X}(n) - \mu}{\sqrt{\frac{S^2(n)}{n}}}$$

- An exact 100(1-α) percent confidence interval for μ is given by

$$\overline{X}(n) \pm t_{n-1,1-\frac{\alpha}{2}} \sqrt{\frac{S^2(n)}{n}}$$

T distribution

- $t_{n-1,1-\frac{\alpha}{2}}$ is the upper critical point for the t distribution with n-1 dof.

- The t distribution is less peaked and has longer tails than the normal distribution, so, for any finite n,

$$t_{n-1,1-\frac{\alpha}{2}} > z_{1-\frac{\alpha}{2}}$$

- In practice, the distribution of the Xi's will rarely be normal, and the confidence interval hence created will also be approximate in terms of coverage.

Hypothesis Testing

- Suppose $X_1, X_2, \ldots\ldots X_n$ are IID random variables
  - We would like to test the null Hypothesis H_0
    - $H_0 : \mu = \mu_0$
    - μ_0 is a fixed hypothesized value for μ
  - Intuitively, we would expect that if, $\left|\overline{X}(n) - \mu_0\right|$ large, H_0 is not likely to be true.
  - But we need a consistent rule also !
    - We need a statistic whose distribution is known when H_0 is true …

Hypothesis Testing

- If H_0 is true, t_n will have a t distribution with n-1 dof

$$t_n = \frac{\overline{X}(n) - \mu}{\sqrt{\frac{S^2(n)}{n}}}$$

- Therefore

$$|t_n| > t_{n-1,1-\frac{\alpha}{2}} \quad \text{reject } \mathrm{H}_0$$

$$|t_n| \le t_{n-1,1-\frac{\alpha}{2}} \quad \text{"accept" } \mathrm{H}_0$$

Hypothesis Testing

- Set of all t_n such that the hypothesis is rejected is called the **critical region**
- What is the probability that **t_n falls in the critical region given H_0 is true ?**

  = α

  Called the level (or size) of the test.

  In general 0.05 or 0.1 chosen

Type I error

- Hypothesis rejected when it is true actually
  - Probability of Type I error = level of the test = α
  - α is chosen by the experimenter
  - Hence it is under control

Type II error

- Hypothesis accepted, when it is indeed false.
  - For a fixed level (α) and sample size n probability of Type II error is β
  - Depends on, what is actually true and may be unknown

Power of the test

- $\delta = 1 - \beta$ called the **power** of the test
  - Which is equal to probability of rejecting H_0 when it is false.
  - Clearly a test with high power is desirable
  - Since the power of a test may be unknown, we say
    - "We fail to reject H_0" instead of saying "We accept H_0" when tn does not lie in critical region
    - Because, we generally do not know with any certainty whether H_0 is true or whether H_0 is false, since our test might not be powerful enough to detect any difference between Ho and what is actually true.

Strong law of large numbers

- Second most important result in probability theory
  - X1, X2, ….. Xn IID random variables with finite mean μ

  - If one performs an infinite number of experiments, each resulting in an $\overline{X}(n)$ and n is sufficiently large, then $\overline{X}(n)$ will be arbitrarily close to μ for almost all the experiments.

$$\overline{X}(n) \rightarrow \mu \text{ as } n \rightarrow \infty$$

Danger of replacing Probability distribution by its mean

- Self Study

Class Test - 2

- Wednesday
  - 18 November, 2009